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Fatou's Lemmas for the Set of Integrable Selections

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In this paper, we prove Fatou-type results for the set of Bochner integrable selections from a set-valued function. Also we obtain some new dominated convergence theorems for the Aumann integral. Our results have applications in Economic Theory.

1. INTRODUCTION

In [5], Pucci and Vitillaro gave two Fatou's Lemmas and a dominated convergence theorem for the integral of a correspondence. In their paper, they assumed that X is a separable reflexive Banach space and (T, Σ, μ) is atomless. In 1986, Khan and Majumdar [4] established an approximate w-Ls version of Fatou's Lemma for a separable Banach space. Khan and Majumdar's version of Fatou's Lemma was generalized independently by Yannelis [6] and Balder [1]. Recently, Yannelis [7] has studied w-Ls and s-Li versions of Fatou's Lemma for the set of integrable selections from a set-valued function. He also proved a corresponding dominated convergence theorem in the sence of Kuratowski–Mosco convergence. His results have useful applications in Economics and Game Theory. In this paper, based on Yanelis's recent work, the w-Li and $\overline{\lim}$ versions of Fatou's Lemmas for the set of Bochner integrable selections are derived. Our w-Li version is new, the latter version is an extension of Fatou's Lemma of Pucci and Vitillaro [5]. Moreover, we define the (w)-limit of a sequence of correspondences and give a corresponding dominated convergence theorem. Finally, using the result of Balder [1] we obtain another dominated convergence theorem without the assumption that limit $\varphi(\cdot)$ is convex valued.

2. THE MAIN THEOREMS

In order to study Fatou-type results for the set of Bochner integrable selections, we introduce the following notations and concepts.

0022-247X/92 $5.00

2^A denotes the set of all nonempty subsets of the set A. $\overline{\text{con}}\, A$ denotes the closed convex hull of A. If X is a Banach space, its dual is the space X^* of all continuous linear functions on X, and if $x^* \in X^*$ and $x \in X$, the value of x^* at x is denoted by $\langle x^*, x\rangle$. Let $\Gamma \subset X^*$, $\sigma(X, \Gamma)$ denotes the weak topology on X generated by Γ. It is known that $\sigma(X, \Gamma)$ is the weakest topology on X in which all the functions belonging to Γ are continuous. The topology $\sigma(X, X^*)$ is called simply the weak topology on X.

If A is a subset of Banach space X, $\text{cl}\, A$ and $\text{w-cl}\, A$ denote the norm closure and weak closure of A, respectively.

Let $\{A_n\}$ be a sequence of nonempty subsets of a Banach space X, we will denote by $\text{w-Ls}\, A_n$, $\text{w-Li}\, A_n$, and $\text{s-Li}\, A_n$ the set of its weak limit superior, weak limit inferior, and strong limit inferior points, respectively; i.e.,

$$\text{w-Ls}\, A_n = \{x \in X: x = \text{w-}\lim_k x_{n_k}, x_{n_k} \in A_{n_k}, k = 1, 2, ...\}$$

$$\text{w-Li}\, A_n = \{x \in X: x = \text{w-}\lim_n x_n, x_n \in A_n, n = 1, 2, ...\}$$

$$\text{s-Li}\, A_n = \{x \in X: x = \text{s-}\lim_n x_n, x_n \in A_n, n = 1, 2, ...\}.$$

Let (T, Σ, μ) be a finite measure space and X be a Banach space. The correspondence $F: T \to 2^X$ is said to be lower measurable if for each norm open subset V of X, the set $\{t \in T: F(t) \cap V \neq \emptyset\} \in \Sigma$. The correspondence $F: T \to 2^X$ is said to have a measurable graph if the $G_F = \{(t, x) \in T \times X: x \in F(t)\}$ belongs to $\Sigma \otimes \beta(X)$, where $\beta(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes product σ-algebra. We know that if X is a separable Banach space, (T, Σ, μ) is a complete finite measure space and $F(\cdot)$ has a measurable graph and is nonempty valued, then there exists a measurable selection for $F(\cdot)$; i.e., there is a μ-measurable function $f: T \to X$ such that $f(t) \in F(t)$ μ—a.e. (see [3] or [7]).

Let (T, Σ, μ) be a finite measure space and X be a Banach space. $L_1(\mu, X)$ denotes the space of equivalence classes of X-valued Bochner integrable functions. It is known that f is Bochner integrable if and only if $f: T \to X$ is μ-measurable and $\int_T \|f(t)\|\, d\mu < \infty$.

The integral of the correspondence $F: T \to 2^X$ is defined by

$$\int_T F(t)\, d\mu = \left\{\int_T f(t)\, d\mu: f \in L_1(\mu, X), f(t) \in F(t)\, \mu\text{—a.e.}\right\}.$$

S_F^1 denotes the set of Bochner integrable selections of the correspondence $F: T \to 2^X$; i.e., $S_F^1 = \{f \in L_1(\mu, X): f(t) \in F(t)\, \mu\text{—a.e.}\}$.

Recall that the correspondence F: $T \to 2^X$ is said to be integrably bounded if there exists a function $g \in L_1(\mu)$ such that

$$\sup\{\|x\|: x \in F(t)\} \leqslant g(t)\,\mu\text{—a.e.}$$

Now we study w-Li version of Fatou's Lemma.

THEOREM 2.1. *Let (T, Σ, μ) be a complete finite measure space and X be a separable Banach space. If F_n: $T \to 2^X$ $(n = 1, 2, \ldots)$ is a sequence of nonempty valued correspondences having a measurable graph, and $F_n(t) \subset F(t)\,\mu$—a.e., where F: $T \to 2^X$ is an integrably bounded, weakly compact, nonempty convex valued correspondence. Then*

$$S^1_{\text{w-Li}\,F_n} \subset \text{w-Li}\,S^1_{F_n}.$$

Proof. Since X is separable, X^* contains a countable subset $\{x_i^*\}$ which is weak* dense in X^*. We can define the metric d on X by

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{|\langle x - y, x_i^* \rangle|}{1 + |\langle x - y, x_i^* \rangle|}.$$

It is clear that d is weaker than the weak topology on X. Let f be in $S^1_{\text{w-Li}\,F_n}$; i.e., $f(t) \in \text{w-Li}\,F_n(t)\,\mu$—a.e. and $f \in L_1(\mu, X)$. First note that $f(t) \in \text{w-Li}\,F_n(t)\,\mu$—a.e. implies that there exists a sequence $\{h_n\}$ such that $\text{w-lim}_n\, h_n(t) = f(t)\,\mu$—a.e., and $h_n(t) \in F_n(t)\,\mu$—a.e., $n = 1, 2, \ldots$. It follows that $d\text{-lim}_n\, h_n(t) = f(t)\,\mu$—a.e. For each n, define the correspondence A_n: $T \to 2^X$ by

$$A_n(t) = \left\{ y \in F_n(t) \colon d(y, f(t)) \leqslant d(f(t), F_n(t)) + \frac{1}{n} \right\}.$$

Since d is weaker than the norm topology on X, obviously, the metric function $d(\cdot, \cdot)$ is continuous in the $X \times X$-norm. Using the same argument used in [7, Proof of Lemma 5.3] we claim that $A_n(\cdot)$ has a measurable graph. Also $A_n(\cdot)$ is nonempty valued, by the Aumann measurable selection theorem there exists a measurable function f_n: $T \to X$ such that $f_n(t) \in A_n(t)\,\mu$—a.e. From the definition of $A_n(\cdot)$ and $d\text{-lim}_n\, h_n(t) = f(t)\,\mu$—a.e., $h_n(t) \in F_n(t)\,\mu$—a.e., $n = 1, 2, \ldots$. We have that $d\text{-lim}_n\, f_n(t) = f(t)\,\mu$—a.e. Because $f_n(t) \in A_n(t) \subset F_n(t) \subset F(t)\,\mu$—a.e. and $F(\cdot)$ is weakly compact valued, we obtain $\text{w-lim}_n\, f_n(t) = f(t)\,\mu$—a.e. Since $F(\cdot)$ is integrably bounded, by the Lebesgue dominated convergence theorem, we obtain that $\int_E |x^* f_n(t) - x^* f(t)|\, d\mu \to 0$, for each $x^* \in X^*$ and each $E \in \Sigma$. Furthermore, for any simple function $\psi = \sum_{i=1}^k x_i^* \chi_E$ we can conclude that $\langle \psi, f_n \rangle = \int_T \langle \psi(t), f_n(t) \rangle\, d\mu \to \int_T \langle \psi(t), f(t) \rangle\, d\mu = \langle \psi, f \rangle$. Let $\Gamma = \{\psi$: ψ is a simple function in $L_1(\mu, X)^*\}$, it is now evident, from what has been proved above, that the sequence $\{f_n\}$ converges to f in the topology $\sigma(L_1(\mu, X), \Gamma)$. We now prove that $\{f_n\}$ converges weakly to f. From the definition of Bochner integrable functions and the Hahn–Banach Theorem it is easily shown that Γ separates the points of $L_1(\mu, X)$. Moreover, Diestel's Theorem [2] tells us that if X is separable and $F\colon T \to 2^X$ is an integrably bounded, weakly compact, nonempty convex valued correspondence, then S_F^1 is a weakly compact subset in $L_1(\mu, X)$. By hypothesis, it is clear that for all n, f_n lies in the weakly compact set S_F^1. Without loss of generality, we can assume that $\{f_n\}$ converges weakly to $g \in L_1(\mu, X)$, hence for each $\psi \in \Gamma \subset L_1(\mu, X)^*$, we have $\psi(f) = \psi(g)$. This shows that $f = g$. The proof of the theorem is completed.

COROLLARY 2.1. *Under the assumptions of Theorem* 2.1, *we have*

$$\int_T \text{w-Li}\, F_n(t)\, d\mu \subset \text{w-Li} \int_T F_n(t)\, d\mu.$$

Let X be a Banach space and $\{A_n\} \subset 2^X \setminus \{\varnothing\}$, we will introduce a new convergence of sets, it is weaker than the Kuratowski-Mosco convergence of sets.

DEFINITION 2.1. Let X be a Banach space and $\{A_n\}$ be a given sequence in $2^X \setminus \{\varnothing\}$. The sequence $\{A_n\}$ is said to be (w)-convergent to A, defnoted by $(\text{w})\text{-lim}_n A_n = A$, if $\text{w-Li}\, A_n = \text{w-Ls}\, A_n = A$.

Using [7, Lemma 5.2] and Theorem 2.1, we can prove the following dominated convergence theorem.

THEOREM 2.2. *Suppose that the assumptions of Theorem* 2.1 *hold. Moreover assume that* $(\text{w})\text{-lim}_n F_n(t) = \varphi(t)\,\mu$—a.e. *and* $\varphi(\cdot)$ *is convex valued. Then* $(\text{w})\text{-lim}_n S_{F_n}^1 = S_\varphi^1$.

Proof. Since X is separable and for each $t \in T$, $F(t)$ is a weakly compact set, it implies that $F(t)$ is metrizable, so

$$\text{w-Ls}\, F_n(t) = \bigcap_{k=1}^{\infty} \text{w-cl}\{F_m(t)\colon m \geqslant k\},$$

this means that $\text{w-Ls}\, F_n(t)$ is closed. In view of [7, Lemma 5.2], we have $\text{w-Ls}\, S_{F_n}^1 \subset S_{\text{w-Ls}\, F_n}^1$. It follows directly from Theorem 2.1 that

$$S_\varphi^1 = S_{\text{w-Li}\, F_n}^1 \subset \text{w-Li}\, S_{F_n}^1 \subset \text{w-Ls}\, S_{F_n}^1 \subset S_{\text{w-Ls}\, F_n}^1 = S_\varphi^1.$$

Consequently, $S_\varphi^1 = \text{w-Li}\, S_{F_n}^1 = \text{w-Ls}\, S_{F_n}^1$. That is, $(\text{w})\text{-lim}_n S_{F_n}^1 = S_\varphi^1$.

In particular, we have

COROLLARY 2.2. *The assumptions of Theorem 2.2 imply that*

$$(\mathrm{w})\text{-}\lim_n \int_T F_n(t)\, d\mu = \int_T \varphi(t)\, d\mu.$$

In [5], the upper and lower limit of a sequence of nonempty subsets in Banach space X were defined by

$$\underline{\lim}_n A_n = \{x \in X: x^*(x) \leqslant \underline{\lim}_n (\sup_{y \in A_n} x^*(y)),\ x^* \in X^*\}$$

and

$$\overline{\lim}_n A_n = \{x \in X: x^*(x) \leqslant \overline{\lim}_n (\sup_{y \in A_n} x^*(y)),\ x^* \in X^*\}.$$

The sequence $\{A_n\}$ is said to be convergent to A in the sense of [5] if $\underline{\lim}_n A_n = \overline{\lim}_n A_n = A$. It is written as (c)-$\lim_n A_n = A$.

It is known that $\underline{\lim}_n A_n$ and $\overline{\lim}_n A_n$ are two closed convex subsets of X. Also it is clear that

$$\text{w-Li } A_n \subset \underline{\lim}_n A_n, \qquad \text{w-Ls } A_n \subset \overline{\lim}_n A_n.$$

In order to prove the following theorem, we need to prove a lemma.

LEMMA 2.1. *Let $\{A_n\}$ be a sequence of nonempty sets of Banach space X, $A_n \subset K$ for all n, where K is a weakly compact subset of X. Then*

$$\overline{\lim}_n A_n = \overline{\mathrm{con}}\ \text{w-Ls } A_n.$$

Proof. In order to see this, we assume, by contradiction, that there exists $x_0 \in \overline{\lim}_n A_n \backslash \overline{\mathrm{con}}$ w-Ls A_n. Thus there is a $x_0^* \in X^*$ such that

$$x_0^*(x_0) > \alpha > \sup_{y \in \overline{\mathrm{con}} \text{ w-Ls } A_n} x_0^*(y). \tag{1}$$

Let $x_n \in A_n$ ($n = 1, 2, ...$). We consider the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_0^*(x_{n_i}) \to a$. Since $\{x_n\} \subset K$ and K is weakly compact, without loss of generality, we can assume that $\{x_{n_i}\}$ converges weakly to x. Thus $x \in$ w-Ls A_n and $x_0^*(x) = \lim_i x_0^*(x_{n_i})$. From (1) we conclude that

$$\begin{aligned} x_0^*(x) &\leqslant \sup_{y \in \text{w-Ls } A_n} x_0^*(y) \\ &\leqslant \sup_{y \in \overline{\mathrm{con}} \text{ w-Ls } A_n} x_0^*(y) < \alpha < x_0^*(x_0). \end{aligned}$$

It follows that

$$\overline{\lim_n}\, x_0^*(x_n) \leqslant \sup_{y \in \text{w-Ls}\, A_n} x_0^*(y) < \alpha < x_0^*(x_0). \tag{2}$$

For each n, take $x_n \in A_n$ such that $x_0^*(x_n) > \sup_{y \in A_n} x_0^*(y) - 1/n$, $n = 1, 2, \ldots$. It is clear that

$$\overline{\lim_n}\, x_0^*(x_n) \geqslant \overline{\lim_n}\, (\sup_{y \in A_n} x_0^*(y)). \tag{3}$$

By (2) and (3) we have

$$\begin{aligned}\overline{\lim_n}\, (\sup_{y \in A_n} x_0^*(y)) &\leqslant \overline{\lim_n}\, x_0^*(x_n)\\ &\leqslant \sup_{y \in \text{w-Ls}\, A_n} x_0^*(y) < \alpha < x_0^*(x_0).\end{aligned}$$

This is a contradiction to $x_0 \in \overline{\lim}_n A_n$.

THEOREM 2.3. *Let (T, Σ, μ) be a finite positive measure space, X be a separable Banach space, and F_n: $T \to 2^X$ $(n = 1, 2, \ldots)$ be a sequence of nonempty valued correspondence such that for all n, $F_n(t) \subset F(t)\, \mu$*—a.e., *where F: $T \to 2^X$ is an integrably bounded, weakly compact, nonempty convex valued correspondence. Then*

$$\overline{\lim_n}\, S^1_{F_n} \subset S^1_{\overline{\lim}_n F_n}.$$

Proof. In [7], Yannelis has shown that

$$\text{w-Ls}\, S^1_{F_n} \subset S^1_{\overline{\text{con}}\, \text{w-Ls}\, F_n}.$$

It is easily seen that $S^1_{\overline{\text{con}}\, \text{w-Ls}\, F_n}$ is a closed convex set in $L_1(\mu, X)$. Hence $\overline{\text{con}}\, \text{w-Ls}\, S^1_{F_n} \subset S^1_{\overline{\text{con}}\, \text{w-Ls}\, F_n}$. Moreover for all n, $S^1_{F_n} \subset S^1_F$, $F_n(t) \subset F(t)\, \mu$ —a.e. and S^1_F, $F(t)$ are weakly compact in $L_1(\mu, X)$, X, respectively. By Lemma 2.1 we conclude that[1]

$$\overline{\lim_n}\, S^1_{F_n} \subset S^1_{\overline{\lim}_n F_n}.$$

Remark 2.1. Theorem 2.3 remains true in arbitrary Banach spaces. The proof proceeds as follows: It should be noted that Diestel's Theorem is true in arbitrary Banach spaces. That is, if X is an arbitrary Banach space and F: $T \to 2^X$ is an integrably bounded, weakly compact, nonempty convex valued correspondence, then S^1_F is a weakly compact set in $L_1(\mu, X)$.

[1] Note that we regard any empty set as a singleton $\{\varnothing\}$, and promise $x^*(\varnothing) = -\infty$, for all $x^* \in X^*$. Further for a sequence of subsets without nonempty assumption, its upper limit ($\overline{\lim}$) is defined. We can prove that Lemma 2.1 remains true.

Indeed, let $\{g_n\}$ be a sequence chosen from S_F^1. Since g_n is measurable, so g_n is essentially separable valued. Without loss of generality we can assume that $g_n(T) \subset X_0$ and $g_n(t) \in F(t)$, $t \in T$ for all n, where X_0 is a separable closed subspace of X. Let $\bar{F}(t) = F(t) \cap X_0$, $t \in T$. Then $\bar{F}$: $T \to 2^{X_0}$ is also an integrably bounded, weakly compact, nonempty convex valued correspondence. In view of Diestel's Theorem, we have that for all n, g_n lies in the weakly compact set $S_{\bar{F}}^1$ of $L_1(\mu, X_0)$. Furthermore, we can claim that S_F^1 is a weakly compact set in $L_1(\mu, X)$. This leads us to conclude that the Banach space X can be taken to be nonseparable in all results (except Theorem 3.2 and Lemma 5.3) which are due to Yannelis [7]. In particular, [7, Lemma 5.1] is true for nonseparable Banach spaces. One can now proceed as in the proof of Theorem 2.3 to complete the argument.

COROLLARY 2.3. *Suppose that the assumptions* (*except the separability of* X) *of Theorem* 2.3 *hold, we have that*

$$\overline{\lim_n} \int_T F_n(t)\, d\mu \subset \int_T \overline{\lim_n}\, F_n(t)\, d\mu.$$

Proof. By the assumption it follows that

$$\overline{\text{con}}\ \text{w-Ls}\ S_{F_n}^1 \subset S^1_{\overline{\text{con}}\ \text{w-Ls}\ F_n}.$$

Define the mapping ψ: $L_1(\mu, X) \to X$ by $\psi(f) = \int_T f(t)\, d\mu$. Then ψ is linear bounded. It can be easily shown that

$$\psi(\overline{\text{con}}\ \text{w-Ls}\ S_{F_n}^1) = \overline{\text{con}}\ \text{w-Ls}\ \psi(S_{F_n}^1) \subset \psi(S^1_{\overline{\text{con}}\ \text{w-Ls}\ F_n}),$$

i.e., $\overline{\text{con}}$ w-Ls $\int_T F_n(t)\, d\mu \subset \int_T \overline{\text{con}}$ w-Ls $F_n(t)\, d\mu$. Using Lemma 2.1, we obtain

$$\overline{\lim_n} \int_T F_n(t)\, d\mu \subset \int_T \overline{\lim_n}\, F_n(t)\, d\mu.$$

Corollary 2.3 generalizes [5, Theorem 3.5(b)] in several directions.

3. The Applications of w-Li Version of Fatou's Lemma

In the statements of the dominated convergence theorems of [5, 7], it should be noted that the limit $\varphi(\cdot)$ of the convergence sequence $\{F_n(\cdot)\}$ is convex valued. In this section, [1, Theorem 2.1] and w-Li version of Fatou's Lemma will be used to prove a new dominated convergence theorem without the condition "limit $\varphi(\cdot)$ of the convergence sequence $\{F_n(\cdot)\}$ is convex valued."

Theorem 2.1 of Balder [1] tells us that if $\{f_k\}$ is a sequence of Bochner integrable functions which satisfies the following conditions:

(i) $f_k(t) \in F(t)\,\mu$—a.e. for all k, where $F(\cdot)$ is an integrably bounded, weakly compact valued correspondence, and $F(\cdot)$ has a measurable graph.

(ii) $\text{w-lim}_k \int_T f_k(t)\,d\mu$ exists. Then for each $\varepsilon > 0$, there is a function $f_\varepsilon \in L_1(\mu, X)$ such that

$$\left\| \int_T f_\varepsilon(t)\,d\mu - \text{w-}\lim_k \int_T f_k(t)\,d\mu \right\| < \varepsilon,$$

$$f_\varepsilon(t) \in \text{w-Ls}\, f_k(t)\,\mu\text{—a.e.}$$

Consequently, for any sequence of Bochner integrable functions which satisfies condition (i), we can conclude that

$$\text{w-Ls} \int_T f_k(t)\,d\mu \subset \text{cl} \int_T \text{w-Ls}\, f_k(t)\,d\mu.$$

For any complete and finite measure space, [1, Theorem 2.1] remains true without the assumption that $F(\cdot)$ has a measurable graph. In fact, it can be shown that $\text{w-Ls}\, f_k(\cdot)$ is lower measurable (or see [7, p. 78]). Without loss of generality, we can assume that X is a separable Banach space. Let $H(t) = \text{w-cl}\{f_k(t)\}$, $t \in T$. Since for all k, $f_k(t) \in F(t)\,\mu$—a.e. and $F(t)$ is a weakly compact set of X, therefore, for any norm open set V of X, we have

$$\{t \in T\colon H(t) \cap V \neq \varnothing\} = \{t \in T\colon \text{w-Ls}\, f_k(t) \cap V \neq \varnothing\} \cup \bigcup_{k=1}^{\infty} \{t \in T\colon f_k(t) \cap V \neq \varnothing\}.$$

By the assumptions, we claim that $\{t \in T\colon H(t) \cap V \neq \varnothing\} \in \Sigma$. This means that $H(\cdot) = \text{w-cl}\{f_k(\cdot)\}$ is lower measurable. Also $H(\cdot)$ is nonempty closed valued, hence $H(\cdot)$ has a measurable graph. Moreover, $H(\cdot)$ is also an integrably bounded, weakly compact valued correspondence. So $F(\cdot)$ can be replaced by $H(\cdot)$ in [1, Theorem 2.1].

THEOREM 3.1. *Suppose that the assumptions of Theorem 2.1 hold. Moreover assume that* $(\text{w})\text{-lim}_n F_n(t) = \varphi(t)$, $t \in T$. *Then*

$$(\text{c})\text{-}\lim_n \int_T F_n(t)\,d\mu = \overline{\text{con}} \int_T \varphi(t)\,d\mu.$$

Proof. From [1, Theorem 2.1], we obtain

$$\text{w-Ls} \int_T F_n(t)\, d\mu \subset \text{cl} \int_T \text{w-Ls}\, F_n(t)\, d\mu,$$

thus

$$\begin{aligned} \overline{\text{con}}\, \text{w-Ls} \int_T F_n(t)\, d\mu &\subset \overline{\text{con}}\, \text{cl} \int_T \text{w-Ls}\, F_n(t)\, d\mu \\ &= \overline{\text{con}} \int_T \text{w-Ls}\, F_n(t)\, d\mu. \end{aligned}$$

Since $(\text{w})\text{-lim}_n F_n(t) = \varphi(t)$, $t \in T$, i.e., $\text{w-Ls}\, F_n(t) = \text{w-Li}\, F_n(t)$, $t \in T$. It follows that

$$\overline{\text{con}}\, \text{w-Ls} \int_T F_n(t)\, d\mu \subset \overline{\text{con}} \int_T \text{w-Li}\, F_n(t)\, d\mu.$$

By Corollary 2.1, we have

$$\begin{aligned} \overline{\text{con}}\, \text{w-Ls} \int_T F_n(t)\, d\mu &\subset \overline{\text{con}}\, \text{w-Li} \int_T F_n(t)\, d\mu \\ &\subset \underline{\lim}_n \int_T F_n(t)\, d\mu. \end{aligned}$$

In view of Lemma 2.1, the inclusion

$$\overline{\lim_n} \int_T F_n(t)\, d\mu \subset \underline{\lim}_n \int_T F_n(t)\, d\mu$$

holds. This shows taht $(\text{c})\text{-lim}_n \int_T F_n(t)\, d\mu = \overline{\text{con}} \int_T \varphi(t)\, d\mu$.

COROLLARY 3.1. *Suppose that the assumptions* (*except the condition* (iii)) *of* [7, *Theorem* 3.2] *hold. We can claim that*

$$(\text{c})\text{-}\lim_n \int_T \varphi_n(t)\, d\mu = \overline{\text{con}} \int_T \varphi(t)\, d\mu.$$

Proof. Since $\{A_n\}$ converges to A in the sense of Kuratowski–Mosco convergence implies $(\text{w})\text{-lim}_n A_n = A$. By Theorem 3.1, we prove the claim.

I am most grateful to the referee for his comments and suggestions which helped us to provide the following application of w-Li version of Fatou's Lemma.

Let (T, Σ, μ) be a finite measure space, X be a Banach space, P be a metric space, and φ: $T \times P \to 2^X$ be a correspondence.

We say that for each $t \in T$, $\varphi(t, \cdot)$ is weak-lower semicontinuous if $\varphi(t, p) \subset$ w-Li $\varphi(t, p_n)$ whenever the sequence $\{p_n\}$ converges (in the metric topology) to p.

Note that the integral of $\varphi(t, p)$ is

$$\int_T \varphi(t, p)\, d\mu(t) = \left\{ \int_T f(t)\, d\mu(t) : f \in S^1_\varphi(p) \right\},$$

where $S^1_\varphi(p) = \{f \in L_1(\mu, X) : f(t) \in \varphi(t, p)\, \mu\text{—a.e.}\}$.

Theorem 2.1 can be used to prove that integration preserves weak-lower semicontinuity.

THEOREM 3.2. *Let (T, Σ, μ) be a complete finite measure space, and X be a separable Banach space. Let φ: $T \times P \to 2^X$ be a nonempty valued correspondence such that*

(i) *For each $t \in T$, $\varphi(t, \cdot)$ is weak-lower semicontinuous,*

(ii) *for all $(t, p) \in T \times P$, $\varphi(t, p) \subset F(t)$, where F: $T \to 2^X$ is an integrably bounded, weakly compact, nonempty valued correspondence, and*

(iii) *for each fixed $p \in P$, $\varphi(\cdot, p)$ has a measurable graph. Then $S^1_\varphi(\cdot)$ is weak-lower semicontinuous.*

In particular, $\int_T \varphi(t, \cdot)\, d\mu(t)$ is weak-lower semicontinuous.

Proof. Let $\{p_n\}$ converge to p in the metric space P. By assumption it follows from Theorem 2.1 that

$$S^1_{\text{w-Li}\, \varphi(\cdot, p_n)} \subset \text{w-Li}\, S^1_{\varphi(\cdot, p_n)} = \text{w-Li}\, S^1_\varphi(p_n).$$

Since $\varphi(t, \cdot)$ is weak-lower semicontinuous for each $t \in T$, we have that $\varphi(t, p) \subset$ w-Li $\varphi(t, p_n)$ for each $t \in T$. Consequently,

$$S^1_\varphi(p) = S^1_{\varphi(\cdot, p)} \subset S^1_{\text{w-Li}\, \varphi(\cdot, p_n)} \subset \text{w-Li}\, S^1_\varphi(p_n).$$

This shows that $S^1_\varphi(\cdot)$ is weak-lower semicontinuous.

Now define the mapping ψ: $L_1(\mu, X) \to X$ by $\psi(f) = \int_T f(t)\, d\mu(t)$. Note that $\psi(\cdot)$ is linear and norm continuous, hence, weakly continuous. It follows that $\psi(S^1_\varphi(p)) \subset \psi(\text{w-Li}\ S^1_\varphi(p_n)) = \text{w-Li}\ \psi(S^1_\varphi(p_n))$; i.e., $\int_T \varphi(t, p)\, d\mu(t) \subset \text{w-Li} \int_T \varphi(t, p_n)\, d\mu(t)$. This means that $\int_T \varphi(t, \cdot)\, d\mu(t)$ is weak-lower semicontinuous.

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